Particle propagation on a circle with a point interaction

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Abstract
We study a particle propagation on a circle in the presence of a point interaction.
We show that the one-particle Feynman kernel can be written as the sum of
reflected and transmitted trajectories which are weighted by the elements of the
nth power of the scattering matrix evaluated on a line with a point interaction.
As a by-product, we find a three-parameter family of trace formulae as a
generalization of the Poisson summation formula.

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1. Introduction

A quantum system restricted on a bounded domain has become more relevant for theoretical
physics. There, the role of boundary conditions is very important not only for the long distance
(infrared) regime but also for the short distance (ultraviolet) regime. As is well known,
due to the presence of boundaries, a nontrivial problem arises with specifying domains of
self-adjointness for observables, typically for the Hamiltonian. Note that the Hamiltonian
operator, which is usually given by a (partial) differential operator, still remains formal
without specifying its self-adjoint domains corresponding to the different boundary conditions.
Mathematically, the correct framework for treating the boundary conditions in quantum theory
is by means of the analysis of von Neumann's self-adjoint extension of the symmetric operator
[1]. (Also see [2] for the dissipative extensions of an abstract symmetric operator with equal
deficiency indices.) Physically speaking, the variety of boundary conditions provided by
the self-adjoint extension of the formal Hamiltonian implies the very rich structure of point
interactions available in quantum theory.

Although in the operator formalism point interactions have been extensively discussed
in the literature, the path-integral description of point interactions has not yet been fully
understood. Mathematically speaking, this is mainly due to the lack of trace formulae suitable
for the point interactions. Physically speaking, on the other hand, this is mainly due to the
lack of our knowledge about the classical trajectories for a particle in the presence of point

interactions. The aim of this paper is to fulfill a gap of the description for boundary conditions between the operator formalism and the path-integral formalism; we would like to propose a physically transparent prescription on how to incorporate the boundary conditions obtained in the operator formalism into the path-integral description. To illustrate our idea in a simple setting, in this paper we will consider one-particle quantum mechanics on a circle in the presence of a single point interaction.

To begin with, let us first consider a quantum particle on a circle of circumference L in the presence of a δ'-interaction described by the formal Hamiltonian $H = -\mathrm{d}^2/\mathrm{d}x^2 + 2c\delta'(x)$, where $c \in \mathbb{R}$ is the dimensionless coupling constant and the prime ($'$) indicates the derivative with respect to x. (Here, as in the following, we use units where $\hbar = 2m = 1$.) It is known that the δ'-interaction belongs to the so-called scale-independent subfamily of point interactions [3] and is verified by the boundary conditions $\psi(L) = [(1 - c)/(1 + c)]\psi(0)$ and $\psi'(L) = [(1 + c)/(1 - c)]\psi'(0)$ [4, 5], where ψ is the square integrable wavefunction on an interval $(0, L)$. Although the Feynman kernel of this system has been analyzed in the literature [6, 7], the physical interpretation for the weight factors (see below) remains open. We would first like to address this issue.

As ubiquitous in the scale-independent point interactions, in this δ'-interaction case the wave numbers are quantized in an integer step so that it is easy to rewrite the Feynman kernel $K(x, T; x_0, 0) = \langle x|\mathrm{e}^{-\mathrm{i}HT}|x_0\rangle$ evaluated in the operator formalism as the path-integral representation with the help of the Poisson summation formula. The resultant kernel takes the form

$$K(x, T; x_0, 0) = \frac{1}{\sqrt{4\pi \mathrm{i}T}} \sum_{n \in \mathbb{Z}} \left\{ \cos(n\theta)\, \mathrm{e}^{\mathrm{i}\frac{T}{4}\left(\frac{nL+x-x_0}{T}\right)^2} \mp \sin(n\theta)\, \mathrm{e}^{\mathrm{i}\frac{T}{4}\left(\frac{(n+1)L-x-x_0}{T}\right)^2} \right\}, \tag{1}$$

where $0 \leqslant \theta := \mathrm{Arccos}[(1 - c^2)/(1 + c^2)] < \pi$ and $-$ (+) sign for $c > 0$ ($c < 0$). Arccos is the principal value of the inverse cosine. Note that the presence of a point interaction breaks the global translational invariance. As a consequence, the kernel (1) is the sum of partial amplitudes for the translational invariant and variant classes of trajectories, which are weighted by the factors $\cos(n\theta)$ and $\sin(n\theta)$ respectively.

Before discussing the physical meaning of the weight factors, we have to reveal the particle propagations described by (1). To this end, it should first be noted that $c = 0$ leads to $\theta = 0$ so that equation (1) becomes the well-known form of the one-particle Feynman kernel on a circle with periodic boundary conditions. As discussed in many textbooks (see for example [8, 9]), in this $c = 0$ case the kernel (1) is the sum of partial amplitudes for transitions via classical paths distinguished by the homotopy class of S^1, i.e. the winding number. For nonzero c, however, the classical trajectories of a particle are not so trivial due to the presence of δ'-potential, which acts as a point scatterer. When a particle reaches the position of the point scatterer, there must be in general two possibilities: reflection or transmission. Thus, the paths for a particle interacting n-times with the point interaction must consist of 2^n distinct paths. As an example, the classical worldlines for $n = 2$ and -2 in (1) are depicted in figure 1. As we will see below, half of these 2^n paths belong to the translational invariant class and the other half to the translational variant class.

Now it is time to discuss the physical meaning of the weight factors. It is intuitively clear that the reflected trajectory should be weighted with a reflection coefficient R_+ (R_-) for every time when a particle is reflected by the point scatterer from left to left (right to right), where R_+ (R_-) is the reflection coefficient for a particle propagating the negative half-line $\mathbb{R}_-$ (positive half-line $\mathbb{R}_+$). Similarly, a transmitted trajectory should be weighted with a transmission coefficient T_+ (T_-) for every time when a particle is transmitted by the point scatterer from left to right (right to left), where T_+ (T_-) is the transmission coefficient for
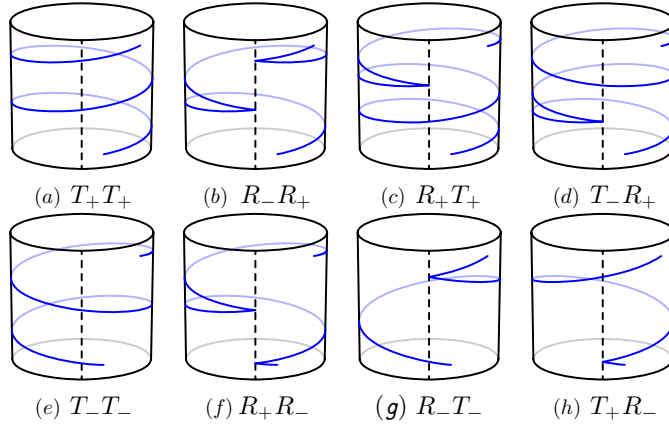
Figure 1. Classical worldlines for a particle scattered twice by the point interaction. Time flows along the vertical direction. The dashed line represents the worldline for the point interaction. In the case of δ'-interaction with $c > 0$, these $2 \times 2^2 = 8$ trajectories are weighted by the factors $T_\pm T_\pm = \cos^2\theta$, $R_\mp R_\pm = -\sin^2\theta$, $R_\pm T_\pm = \mp\sin\theta\cos\theta$ and $T_\mp R_\pm = \mp\cos\theta\sin\theta$.

a particle propagating from $\mathbb{R}_-$ to $\mathbb{R}_+$ and vice versa. The physical meaning of the weight factors is now obvious: these must be elements of the nth power of the one-particle scattering matrix $S^{(1)}$, which we would like to call the *n-times scattering matrix* $S^{(n)}$, evaluated on a line with a point interaction at the origin (see section 3). In the case of δ'-interaction, it is easy to compute the one-particle scattering matrix. The result is

$$S^{(1)} = \begin{pmatrix} T_+ & R_- \\ R_+ & T_- \end{pmatrix} = \begin{cases} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, & \text{for} \quad c > 0, \\ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, & \text{for} \quad c < 0, \end{cases} \tag{2}$$

which is just the rotational matrix. Thus, the n-times scattering matrix is given by just replacing the argument θ in (2) with $n\theta$. These matrix elements are nothing but the weight factors in (1). In this sense, the identity $\cos^2(n\theta) + \sin^2(n\theta) = 1$ is a consequence of the unitarity of the scattering matrix and can be viewed as the *partial amplitude unitarity*.

So far, we have studied only the case of δ'-interaction; it seems that the above discussion is valid for any one-particle quantum mechanics on a circle with a single point interaction. As we will show in the rest of this paper, this observation is indeed true. Now it is time to give an explicit statement for the purpose of this paper. The main goal of this paper is to show the following statement: *the Feynman kernel for a spinless particle moving freely on a circle of circumference L with a single point interaction at the origin can be written in the following generic form*:

$$K(x, T; x_0, 0) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \, \mathrm{e}^{\mathrm{i}T[p(\frac{x-x_0}{T}) - p^2]}$$

$$+ \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \left\{ S_{++}^{(n)}(p) \, \mathrm{e}^{\mathrm{i}T[p(\frac{nL+x-x_0}{T}) - p^2]} + S_{-+}^{(n)}(p) \, \mathrm{e}^{\mathrm{i}T[p(\frac{(n+1)L-x-x_0}{T}) - p^2]} \right\}$$

$$+ \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \left\{ S_{--}^{(n)}(p) \, \mathrm{e}^{\mathrm{i}T[-p(\frac{-nL+x-x_0}{T}) - p^2]} + S_{+-}^{(n)}(p) \, \mathrm{e}^{\mathrm{i}T[-p(\frac{(-n+1)L-x-x_0}{T}) - p^2]} \right\}$$

$$+ (bound\ state\ contribution), \tag{3}$$

where $S_{\pm\pm}^{(n)}$ and $S_{\pm\mp}^{(n)}$ are the elements of the n-times scattering matrix.

This statement is based on the following observations.

(i) The classical trajectories $x_{cl}(t)$ for a particle propagating from $(x_0, 0)$ to (x, T) scattered n-times by the point interaction are exhausted by $x_{cl}(t) = x_0 + v_{cl}t$, where $v_{cl} = (\pm nL + x - x_0)/T$ for a translational invariant class and $v_{cl} = ((\pm n + 1)L - x - x_0)/T$ for a translational variant class of classical trajectories, where the '+' sign is for the trajectory of the right-moving outgoing particle and the '−' sign is for that of the left-moving outgoing particle.

(ii) Any paths for a particle traveling from $(x_0, 0)$ to (x, T) with momentum p are categorized into four cases, that is, the propagation from left to right, from left to left, from right to left and from right to right. The corresponding plane waves are $e^{ip(nL+x-x_0)}$, $e^{ip((n+1)L-x-x_0)}$, $e^{-ip(-nL+x-x_0)}$ and $e^{-ip((-n+1)L-x-x_0)}$, respectively. These four classes of classical trajectories should be weighted by the factors $S_{++}^{(n)}(p)$, $S_{-+}^{(n)}(p)$, $S_{--}^{(n)}(p)$ and $S_{+-}^{(n)}(p)$, which are the elements of the n-times scattering matrix $S^{(n)}(p)$ on the basis of right- and left-moving momentum modes (see section 3).

(iii) The bound-state contribution, even if it exists, does not affect the scattering process on a line such that it can be added at the end of computation.

The purpose of this paper is to show the validity of (3) for point interactions allowed in quantum mechanics, which can be classified, as mentioned before, by means of the analysis of the self-adjoint extension of the formal Hamiltonian operator. In physical language, the self-adjoint extension of the formal Hamiltonian is translated into the requirement for the global conservation of the probability current density $j(0) = j(L)$, where $j = -i((\psi^*)'\psi - \psi^*\psi')$ with ψ being the wavefunction on the Hilbert space consisting of square integrable functions on the interval $(0, L)$. The quantum mechanical system for a free particle on a circle is known to admit a $U(2)$ family of distinct point interactions characterized by the boundary conditions [6, 7]

$$(U - \mathbb{1})\vec{\Psi}(0_+) + iL_0(U + \mathbb{1})\vec{\Psi}'(0_+) = \vec{0}, \tag{4}$$

where

$$\vec{\Psi}(x) := \begin{pmatrix} \psi(x) \\ \psi(L-x) \end{pmatrix}, \qquad \vec{\Psi}'(x) = \begin{pmatrix} \psi'(x) \\ -\psi'(L-x) \end{pmatrix}, \qquad 0 < x < L. \tag{5}$$

U is a 2×2 unitary matrix and L_0 is an arbitrary real constant length scale, which is just introduced to adjust the length dimension of the equation. For the following discussions, it is convenient to parameterize the matrix $U \in U(2)$ as the following spectral decomposition form:

$$U = e^{i\alpha_+}P_+ + e^{i\alpha_-}P_-, \tag{6}$$

$$P_\pm = \frac{\mathbb{1} \pm \vec{e} \cdot \vec{\sigma}}{2}, \tag{7}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector of the Pauli matrices, $e^{i\alpha_\pm}$ ($0 \leqslant \alpha_\pm < 2\pi$) are the two eigenvalues of the unitary matrix U and $P_\pm$ is the corresponding projection operator fulfilling $P_+ + P_- = \mathbb{1}$, $(P_\pm)^2 = P_\pm$ and $P_\pm P_\mp = 0$. $\vec{e} = (e_x, e_y, e_z)$ is a real unit vector satisfying the condition $e_x^2 + e_y^2 + e_z^2 = 1$. In this paper, we derive analytical forms of the one-particle Feynman kernel with these parameters.

It is worthwhile to point out here that if we multiply the projection operators $P_\pm$ with (4) on the left, the boundary conditions boil down to the following two independent equations:

$$P_\pm[\vec{\Psi}(0_+) + L_\pm\vec{\Psi}'(0_+)] = \vec{0}, \tag{8}$$

where

$$L_\pm := L_0 \cot(\alpha_\pm/2). \tag{9}$$

It should be noted that (8) is not well defined when $\alpha_\pm = 0$. We will, however, use (8) instead of (4) as the boundary conditions by taking a careful limit for the case of $\alpha_\pm = 0$.

The rest of this paper is organized as follows. In section 2, we derive the general forms of the reflection and transmission coefficients for a particle on a whole line in the presence of a point interaction at the origin. In section 3, we define the one-particle scattering matrix on $\mathbb{R}\backslash\{0\}$ and then introduce the n-times scattering matrix. Section 4 is devoted to detailed analysis of the spectral property for the free Hamiltonian (i.e. Laplace operator) on $S^1\backslash\{0\}$. As a by-product, we find a three-parameter family of trace formulae which provide a direct connection between the quantum energy spectrum and classical length spectrum of S^1 with a point singularity. These can be regarded as generalizations of the Poisson summation formula. In section 5 we give a proof of (3). We conclude in section 6.

2. Reflection and transmission coefficients on $\mathbb{R}\backslash\{0\}$

In this section we will calculate the matrix elements of the scattering matrix, that is, the reflection and transmission coefficients for a continuum state once scattered by the point interaction at the origin on a whole line.

The reflection and transmission coefficients for right- and left-moving incidental waves with momentum $k > 0$ are given by

$$\psi_{k,+}(x) = \begin{cases} e^{ikx} + R_+(k)\, e^{-ikx}, & \text{for} \quad x < 0, \\ T_+(k)\, e^{ikx}, & \text{for} \quad x > 0, \end{cases} \tag{10a}$$

and

$$\psi_{k,-}(x) = \begin{cases} T_-(k)\, e^{-ikx}, & \text{for} \quad x < 0, \\ e^{-ikx} + R_-(k)\, e^{ikx}, & \text{for} \quad x > 0. \end{cases} \tag{10b}$$

Point interactions consistent with the probability conservation $j(0_+) = j(0_-)$ are characterized by the same boundary conditions as (8) but with the two-component vectors

$$\vec{\Psi}(x) = \begin{pmatrix} \psi(x) \\ \psi(-x) \end{pmatrix}, \qquad \vec{\Psi}'(x) = \begin{pmatrix} \psi'(x) \\ -\psi'(-x) \end{pmatrix}, \qquad 0 < x < \infty. \tag{11}$$

Plugging (10a) and (10b) into the boundary conditions (8) with (11), we get the matrix equations

$$P_\pm Z(k) = e^{i\delta_\pm(k)} P_\pm, \tag{12}$$

where

$$Z(k) := \begin{pmatrix} R_-(k) & T_+(k) \\ T_-(k) & R_+(k) \end{pmatrix}, \tag{13}$$

and

$$0 \leqslant \delta_\pm(k) := 2\mathrm{Arccot}(kL_\pm) = \frac{1}{i}\mathrm{Log}\left(\frac{ikL_\pm - 1}{ikL_\pm + 1}\right) < 2\pi. \tag{14}$$

Arccot and Log are the principal values of the inverse cotangent and logarithm, respectively. Now it is easy to find the reflection and transmission coefficients. Equation (12) implies that the matrix $Z(k)$ is unitary and has the spectral decomposition $Z(k) = e^{i\delta_+(k)} P_+ + e^{i\delta_-(k)} P_-$. Thus,

$$Z(k) = \frac{e^{i\delta_+(k)} + e^{i\delta_-(k)}}{2}\mathbb{1} + \frac{e^{i\delta_+(k)} - e^{i\delta_-(k)}}{2}\vec{e}\cdot\vec{\sigma}, \tag{15}$$

from which we find

$$R_\pm(k) = \frac{e^{i\delta_+(k)} + e^{i\delta_-(k)}}{2} \mp \frac{e^{i\delta_+(k)} - e^{i\delta_-(k)}}{2} e_z,$$ (16a)

$$T_\pm(k) = \frac{e^{i\delta_+(k)} - e^{i\delta_-(k)}}{2}(e_x \mp i e_y).$$ (16b)

These results are consistent with those obtained in [10, 11] with suitable redefinitions of the parameters.

Several remarks are now in order.

(i) The two eigenvalues of $Z(k)$ satisfy the relations $e^{i\delta_\pm(-k)} = e^{-i\delta_\pm(k)}$, from which we define

$$\delta_\pm(-k) = \begin{cases} 2\pi - \delta_\pm(k) & \text{for} \quad \alpha_\pm \neq 0, \\ -\delta_\pm(k) & \text{for} \quad \alpha_\pm = 0. \end{cases}$$ (17)

Note that this definition ensures the continuity of $\delta_\pm(k)$ $(-\infty < k < \infty)$ at $k = 0$.

(ii) The *phase shifts* $\delta_\pm(k)$ satisfy the following functional identities:

$$\delta'_\pm(k) = -\frac{\sin \delta_\pm(k)}{k},$$ (18)

where the prime (ʹ) indicates the derivative with respect to k. These identities will be important for the proof of (3).

(iii) The reflection and transmission coefficients satisfy

$$[R_\pm(k)]^* = R_\pm(-k), \quad [T_\pm(k)]^* = T_\mp(-k),$$ (19)

where $*$ indicates the complex conjugation.

(iv) In terms of the reflection and transmission coefficients, the unitarity conditions of the matrix $Z(k)$ read as

$$T_\mp(-k)T_\pm(k) + R_\pm(-k)R_\pm(k) = 1,$$ (20a)

$$T_\mp(-k)R_\mp(k) + R_\pm(-k)T_\mp(k) = 0.$$ (20b)

3. Scattering matrix on $\mathbb{R}\backslash\{0\}$

In this section we will first introduce the one-particle scattering matrix (S-matrix) and then define the n-times scattering matrix, whose elements give the weight factors of the Feynman kernel for the contributions scattered n-times by the point interaction.

Let us first define the one-particle S-matrix $S^{(1)}$ on a whole line in the presence of a single point interaction at the origin. On the basis of right- and left-moving momentum modes $\{|+k\rangle, |-k\rangle | k > 0\}$, where $\langle x| \pm k\rangle = e^{\pm ikx}$, the one-particle S-matrix is defined as follows:

$$S^{(1)}(k) = \begin{pmatrix} S^{(1)}_{++}(k) & S^{(1)}_{+-}(k) \\ S^{(1)}_{-+}(k) & S^{(1)}_{--}(k) \end{pmatrix} := \begin{pmatrix} T_+(k) & R_-(k) \\ R_+(k) & T_-(k) \end{pmatrix},$$ (21)

whose matrix elements are graphically represented in figure 2. Noting that the S-matrix can be written as $S^{(1)}(k) = Z(k)\sigma_1$ and $Z(k)$ is unitary, we see that $S^{(1)}(k)$ clearly satisfies the unitarity conditions $[S^{(1)}(k)]^\dagger S^{(1)}(k) = \mathbb{1} = S^{(1)}(k)[S^{(1)}(k)]^\dagger$, which are nothing but a consequence of the probability conservation $j(0_+) = j(0_-)$. For the following discussions, it
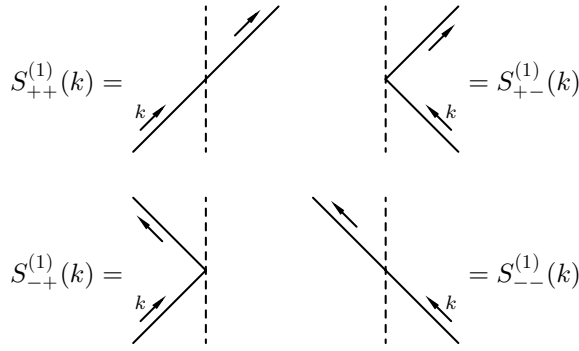
Figure 2. One-particle scattering from a point interaction (point scatterer). Time flows along the vertical direction. The dashed line represents the worldline for the point scatterer. The arrows represent the direction of momentum flow.

is convenient to rewrite the *S*-matrix into the spectral decomposition form:

$$S^{(1)}(k) = s_+(k)\mathcal{P}_+(k) + s_-(k)\mathcal{P}_-(k), \tag{22}$$

where $s_\pm(k)$ are the two eigenvalues of $S^{(1)}(k)$ given by

$$s_\pm(k) = e^{i(\Delta_+(k)+\pi/2)}\left[e_x \sin\Delta_-(k) \pm i\sqrt{1 - e_x^2 \sin^2\Delta_-(k)}\right], \tag{23a}$$

$$\Delta_\pm(k) := \frac{\delta_+(k) \pm \delta_-(k)}{2}, \tag{23b}$$

and $\mathcal{P}_\pm(k)$ are the corresponding projection operators constructed as follows:

$$\mathcal{P}_\pm(k) = \frac{S^{(1)}(k) - s_\mp(k)\mathbb{1}}{s_\pm(k) - s_\mp(k)} = \frac{\mathbb{1} \pm \vec{\varepsilon}(k)\cdot\vec{\sigma}}{2}, \tag{24}$$

where $\vec{\varepsilon}(k) = {}^t(\varepsilon_x(k), \varepsilon_y(k), \varepsilon_z(k))$ is a real unit vector defined as

$$\vec{\varepsilon}(k) := \frac{1}{\sqrt{1 - e_x^2 \sin^2\Delta_-(k)}}\begin{pmatrix} -\cos\Delta_-(k) \\ e_z \sin\Delta_-(k) \\ -e_y \sin\Delta_-(k) \end{pmatrix}. \tag{25}$$

Note that these projection operators $\mathcal{P}_\pm(k)$ satisfy the relations $\mathcal{P}_+(k)+\mathcal{P}_-(k) = \mathbb{1}$, $[\mathcal{P}_\pm(k)]^2 = \mathcal{P}_\pm(k)$ and $\mathcal{P}_\pm(k)\mathcal{P}_\mp(k) = 0$ and that $s_\pm(k)$ satisfy the relations

$$s_\pm(-k) = \begin{cases} 1/s_\pm(k) & \text{for} \quad \alpha_\pm \neq 0 \quad \text{or} \quad \alpha_\pm = 0, \\ 1/s_\mp(k) & \text{for} \quad \alpha_+ = 0, \quad \alpha_- \neq 0 \quad \text{or} \quad \alpha_+ \neq 0, \quad \alpha_- = 0. \end{cases} \tag{26}$$

Next, introduce the *n*-times scattering matrix $S^{(n)}$ as the *n*th power of $S^{(1)}$:

$$S^{(n)}(k) = \begin{pmatrix} S_{++}^{(n)}(k) & S_{+-}^{(n)}(k) \\ S_{-+}^{(n)}(k) & S_{--}^{(n)}(k) \end{pmatrix} := [S^{(1)}(k)]^n. \tag{27}$$

By construction, it is obvious that the *n*-times scattering matrix $S^{(n)}(k)$ satisfies the unitarity conditions $[S^{(n)}(k)]^\dagger S^{(n)}(k) = \mathbb{1} = S^{(n)}(k)[S^{(n)}(k)]^\dagger$, which lead the partial amplitude unitarity of the Feynman kernel. Thanks to the spectral decomposition (22), the *n*-times scattering

matrix is easily computed with the result

$$S^{(n)}(k) = [s_+(k)]^n \mathcal{P}_+(k) + [s_-(k)]^n \mathcal{P}_-(k). \tag{28}$$

Although in the following discussions we do not need the explicit expression for $S^{(n)}$, it may be instructive to write down its matrix elements. A straightforward calculation yields

$$S_{\pm\pm}^{(n)} = e^{in(\Delta_+ + \pi/2)}[T_n(e_x \sin \Delta_-) \mp i e_y \sin \Delta_- \cdot U_{n-1}(e_x \sin \Delta_-)], \tag{29a}$$

$$S_{\mp\pm}^{(n)} = e^{in(\Delta_+ + \pi/2)}[\mp e_z \sin \Delta_- - i \cos \Delta_-] U_{n-1}(e_x \sin \Delta_-), \tag{29b}$$

where T_n and U_n are the Chebyshev polynomials of the first and second kinds, respectively, and satisfy the following relations:

$$T_n(\cos \theta) = \cos(n\theta), \qquad U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}, \qquad n = 0, 1, 2, \ldots. \tag{30}$$

4. Spectrum of $S^1 \backslash \{0\}$

Let us next study the spectrum of the quantum system for a particle on a circle in the presence of a point interaction described by the boundary conditions (4). Although the spectral property of the system has been already studied in the literature [6, 7], these results are not suitable for the purpose of this paper. In this section, we will uncover an amazing relation between the scattering theory on $\mathbb{R} \backslash \{0\}$ discussed in the previous section and positive energy spectrum of $S^1 \backslash \{0\}$. We also derive the trace formulae for the free Hamiltonian (Laplace operator) on $S^1 \backslash \{0\}$.

The general solution to the Schrödinger equation $-\mathrm{d}^2\psi/\mathrm{d}x^2 = E\psi$ on $S^1 \backslash \{0\}$ for positive energy $E = k^2 > 0$ is given by

$$\psi_k(x) = A(k) e^{ikx} + B(k) e^{ik(L-x)}, \qquad k > 0, \tag{31}$$

where the phase factor e^{ikL} in the second term is introduced for later convenience. Note that the two coefficients $A(k)$ and $B(k)$ may depend on k. The general solution for negative energy $E = -\kappa^2 < 0$ will be obtained by just replacing k with $i\kappa$ in (31). We have to be, however, careful about zero energy solutions with $E = 0$, which are *not* necessarily obtained by the naive limit $k \to 0$ in (31): the general solution for $E = 0$ is a first degree polynomial and takes the form

$$\psi_0(x) = A_0 + B_0 x. \tag{32}$$

In this paper, we call the above solution with $B_0 \neq 0$, as well as negative energy solutions, *bound states*. It turns out that any knowledge about these bound states is not necessary for the following discussions. Note that a zero energy solution with the limit $k \to 0$ in (31) is ambiguous because the two terms in (31) are not independent of each other when $k = 0$. This issue will be discussed later.

Substituting (31) into (8), we get the two independent conditions

$$P_\pm(e^{-ikL}\mathbb{1} - e^{i\delta_\pm(k)}\sigma_1)\begin{pmatrix} A(k) \\ B(k) \end{pmatrix} = \vec{0}. \tag{33}$$

Since these two equations are orthogonal to each other, they can be combined into the following form:

$$S^{(1)}(k)\begin{pmatrix} A(k) \\ B(k) \end{pmatrix} = e^{-ikL}\begin{pmatrix} A(k) \\ B(k) \end{pmatrix}, \tag{34}$$

which follows from $P_+ + P_- = \mathbb{1}$ and $S^{(1)} = (e^{i\delta_+} P_+ + e^{i\delta_-} P_-)\sigma_1$. This eigenvalue equation indicates that the positive energy spectrum and eigenfunctions of single particle quantum mechanics on $S^1\backslash\{0\}$ is completely determined by the one-particle S-matrix on $\mathbb{R}\backslash\{0\}$.

In the following, we will analyze the eigenvalue equation (34) in detail.

4.1. Spectrum quantization conditions

Let us first study the spectral property of $S^1\backslash\{0\}$. For non-vanishing $A(k)$ and $B(k)$, we have to implement the following condition:

$$\det[S^{(1)}(k) - e^{-ikL}\mathbb{1}] = 0, \qquad k > 0, \tag{35}$$

which has two branches

$$e^{-ikL} = s_+(k) \qquad \text{and} \qquad e^{-ikL} = s_-(k), \tag{36}$$

where $s_\pm(k)$ are given in (23a). It should be pointed out that these types of equations are commonly referred to as the Bethe ansatz equations. Indeed, the generalization to the n-particle system has been studied in the literature [11] under the name of impurity Bethe ansatz equation.

The positive energy spectrum is determined as the positive roots of the equations

$$f_\pm(k) = 0, \tag{37}$$

where

$$f_\pm(k) := kL + \frac{1}{i} \log s_\pm(k). \tag{38}$$

It should be emphasized that we thought the logarithm function to be the multivalued function defined as $\log z = \{\ln|z| + i \operatorname{Arg} z + i2m\pi \mid 0 \leqslant \operatorname{Arg} z < 2\pi, m \in \mathbb{Z}\}$, where $\operatorname{Arg} z$ is the principal value of the argument. Each integer m determines the branch of the logarithm function and $m = 0$ corresponds to the principal branch.

Note that if $\lim_{k\to 0} s_\pm(k) = 1$, equation (37) may have zero energy solutions. However, the existence of such a zero energy solution does not necessarily imply a physical state in the spectrum because the $k = 0$ solution in (31) becomes trivial and should be thrown away if $A(0) + B(0) = 0$, even though both $A(0)$ and $B(0)$ are not identically zero. Nevertheless, it turns out that such a fake solution is necessary in the trace formulae discussed in the following subsection and the proof of (3).

4.2. Trace formulae

In order to fulfill the gap between the operator formalism and the path-integral formalism, we have to establish the trace formulae for $S^1\backslash\{0\}$. To this end, let us consider delta functions $\delta(f_\pm(k))$. Since the values assumed by $f_\pm(k)$ are $kL + \operatorname{Arg}[s_\pm(k)] + 2m\pi$ for all integers m, the delta functions $\delta(f_\pm(k))$ are periodic functions of $f_\pm$ with a period 2π so that it can be expanded into the Fourier series

$$\delta(f_\pm(k)) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{inf_\pm(k)}. \tag{39}$$

Note that the left-hand side can be written as $\sum_{k_m^\pm\in\sigma_\pm} \left(1/\left|f_\pm'\left(k_m^\pm\right)\right|\right) \delta\left(k - k_m^\pm\right)$, where $\sigma_\pm$ are the sets of both positive and negative roots of the equations $f_\pm(k) = 0$ defined as

$$\sigma_\pm := \left\{ k_m^\pm \in \mathbb{R} \mid f_\pm\left(k_m^\pm\right) = 0, \cdots < k_{-2}^\pm < k_{-1}^\pm < k_0^\pm = 0 < k_1^\pm < k_2^\pm < \cdots \right\}. \tag{40}$$

Equation (40) requires more detailed explanations.

- The negative roots ($m < 0$) are related to the positive ones as follows:

$$k_{-m}^{\pm} = \begin{cases} -k_m^{\pm} & \text{for} \quad \alpha_{\pm} \neq 0 \quad \text{or} \quad \alpha_{\pm} = 0, \\ -k_m^{\mp} & \text{for} \quad \alpha_+ = 0, \quad \alpha_- \neq 0 \quad \text{or} \quad \alpha_+ \neq 0, \quad \alpha_- = 0, \end{cases} \tag{41}$$

  which follow from (26) and (36). These relations will be used in the proof of (3).
- The $k_0^{\pm} = 0$ roots appear only in the following four cases:

$$k_0^+ \qquad \text{for} \quad \alpha_{\pm} \neq 0, \tag{42a}$$

$$k_0^{\pm} \qquad \text{for} \quad \alpha_+ = 0, \quad \alpha_- \neq 0, \quad e_x = 1, \tag{42b}$$

$$k_0^{\pm} \qquad \text{for} \quad \alpha_+ \neq 0, \quad \alpha_- = 0, \quad e_x = -1, \tag{42c}$$

$$k_0^- \qquad \text{for} \quad \alpha_{\pm} = 0. \tag{42d}$$

It turns out that the solution of $k_0^+ = 0$ for $\alpha_{\pm} \neq 0$ and one of the two $k_0^{\pm} = 0$ solutions for $\alpha_+ = 0, \alpha_- \neq 0, e_x = 1$ or $\alpha_+ \neq -, \alpha_- = 0, e_x = -1$ are fake solutions with $A(0) + B(0) = 0$, as explained in the previous subsection. It is emphasized that the $k_0^{\pm} = 0$ solutions (if they exist) must be included in $\sigma_{\pm}$, irrespective of a fake or genuine zero mode.

We note that these remarks will be important for the proof of (3); however, they are not relevant for the rest of this subsection.

Now, identity (39) becomes the following three-parameter family of the trace formulae:

$$\sum_{k_m^{\pm} \in \sigma_{\pm}} \frac{1}{\left| f_{\pm}'\left(k_m^{\pm}\right) \right|} \delta\left(k - k_m^{\pm}\right) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{\mathrm{i} n f_{\pm}(k)}. \tag{43}$$

Note that the derivatives of $f_{\pm}(k)$ are given as follows:

$$f_{\pm}'(k) = L + \frac{\delta_+'(k) + \delta_-'(k)}{2} \pm \frac{\delta_+'(k) - \delta_-'(k)}{2} \frac{-e_x \cos \Delta_-(k)}{\sqrt{1 - e_x^2 \sin^2 \Delta_-(k)}} \tag{44}$$

and satisfy

$$f_{\pm}'(-k) = \begin{cases} f_{\pm}'(k) & \text{for} \quad \alpha_{\pm} \neq 0 \quad \text{or} \quad \alpha_{\pm} = 0, \\ f_{\mp}'(k) & \text{for} \quad \alpha_+ = 0, \quad \alpha_- \neq 0 \quad \text{or} \quad \alpha_+ \neq 0, \quad \alpha_- = 0. \end{cases} \tag{45}$$

As we will see in section 4.4, $f_{\pm}'\left(k_m^{\pm}\right)$ give the normalization factors for the positive energy eigenfunctions.

Before closing this subsection, we try to rewrite formulae (43) in a more practically convenient expression. Since $\sum_{k_m^{\pm} \in \sigma_{\pm}} \left(1 / \left| f_{\pm}'\left(k_m^{\pm}\right) \right|\right) \delta\left(k - k_m^{\pm}\right) = \left(1 / |f_{\pm}'(k)|\right) \sum_{k_m^{\pm} \in \sigma_{\pm}} \delta\left(k - k_m^{\pm}\right)$, (43) can be rewritten into the form $\sum_{k_m^{\pm} \in \sigma_{\pm}} \delta\left(k - k_m^{\pm}\right) = (1/2\pi)|f_{\pm}'(k)| \sum_{n \in \mathbb{Z}} e^{\mathrm{i} n f_{\pm}(k)}$. Thus, by multiplying a smooth test function $F(k)$ and integrating out over the range $-\infty < k < \infty$, the trace formulae (43) can be cast into the following form:

$$\sum_{k_m^{\pm} \in \sigma_{\pm}} F\left(k_m^{\pm}\right) = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \left| \frac{\mathrm{d} f_{\pm}(k)}{\mathrm{d}k} \right| F(k) e^{\mathrm{i} n f_{\pm}(k)}. \tag{46}$$

This identity will be useful for computations of the Casimir energy or the perturbative loop calculations of Feynman diagrams in quantum field theory with nontrivial extended defects (branes or boundaries).

We should make a comment here. Formula (46) reduces to the Poisson summation formula when $\mathrm{d} f_{\pm}(k)/\mathrm{d}k = \text{const}$, which can be realized for a certain region of the parameter space spanned by α_+, α_- and e_x. Then the infinite sequence of the solutions $k_m^{\pm}$ in (46) becomes

equidistant, i.e. $k_m^{\pm} = (2\pi m + \theta)/L$ for $0 \leqslant \theta < 2\pi$. In the $U(2)$ parameter family, however, $\mathrm{d}f_{\pm}(k)/\mathrm{d}k$ can, in general, depend on k and the solutions $k_m^{\pm}$ will be given by transcendental roots with no analytic expression. It should be emphasized that even in this case, formula (46) can be used to derive the Feynman kernel (3).

4.3. Reconstruction of the S-matrix

Once given the two eigenvalues $s_{\pm}(k)$ and the corresponding complete orthonormal eigenvectors $|\pm\rangle := {}^t(A^{\pm}(k), B^{\pm}(k))$ satisfying $\langle \pm | \pm \rangle = 1$, $\langle \pm | \mp \rangle = 0$ and $|+\rangle\langle+|+|-\rangle\langle-| = \mathbb{1}$, the one-particle S-matrix can be reconstructed in terms of $A^{\pm}(k)$ and $B^{\pm}(k)$ from the projection operators

$$\mathcal{P}_{\pm}(k) = |\pm\rangle\langle\pm| = \begin{pmatrix} |A^{\pm}(k)|^2 & A^{\pm}(k)[B^{\pm}(k)]^* \\ [A^{\pm}(k)]^* B^{\pm}(k) & |B^{\pm}(k)|^2 \end{pmatrix}, \tag{47}$$

which of course satisfy $\mathcal{P}_+(k) + \mathcal{P}_-(k) = \mathbb{1}$, $[\mathcal{P}_{\pm}(k)]^2 = \mathcal{P}_{\pm}(k)$ and $\mathcal{P}_{\pm}(k)\mathcal{P}_{\mp}(k) = 0$. It follows from the orthonormality and completeness that

$$|A^+(k)|^2 + |A^-(k)|^2 = 1, \tag{48a}$$

$$|B^+(k)|^2 + |B^-(k)|^2 = 1, \tag{48b}$$

$$A^+(k)[B^+(k)]^* + A^-(k)[B^-(k)]^* = 0, \tag{48c}$$

$$[A^+(k)]^* B^+(k) + [A^-(k)]^* B^-(k) = 0. \tag{48d}$$

By comparing (47) to (24), we find the following relations:

$$|B^{\pm}(-k)|^2 = \begin{cases} |A^{\pm}(k)|^2 & \text{for} \quad \alpha_{\pm} \neq 0 \quad \text{or} \quad \alpha_{\pm} = 0, \\ |A^{\mp}(k)|^2 & \text{for} \quad \alpha_+ = 0, \quad \alpha_- \neq 0 \quad \text{or} \quad \alpha_+ \neq 0, \quad \alpha_- = 0, \end{cases} \tag{49a}$$

$$A^{\pm}(-k)[B^{\pm}(-k)]^* = \begin{cases} [A^{\pm}(k)]^* B^{\pm}(k) & \text{for} \quad \alpha_{\pm} \neq 0 \quad \text{or} \quad \alpha_{\pm} = 0, \\ [A^{\mp}(k)]^* B^{\mp}(k) & \text{for} \quad \alpha_+ = 0, \quad \alpha_- \neq 0 \quad \text{or} \quad \alpha_+ \neq 0, \quad \alpha_- = 0. \end{cases} \tag{49b}$$

Each component of the n-times scattering matrix $S^{(n)}(k)$ in (26) is found to be

$$S_{++}^{(n)}(k) = \sum_{\xi=\pm} [s_{\xi}(k)]^n |A^{\xi}(k)|^2, \tag{50a}$$

$$S_{--}^{(n)}(k) = \sum_{\xi=\pm} [s_{\xi}(k)]^n |B^{\xi}(k)|^2, \tag{50b}$$

$$S_{-+}^{(n)}(k) = \sum_{\xi=\pm} [s_{\xi}(k)]^n [A^{\xi}(k)]^* B^{\xi}(k), \tag{50c}$$

$$S_{+-}^{(n)}(k) = \sum_{\xi=\pm} [s_{\xi}(k)]^n A^{\xi}(k)[B^{\xi}(k)]^*. \tag{50d}$$

4.4. Eigenfunctions

In terms of the orthonormal eigenvectors $|\pm\rangle = {}^t(A^{\pm}(k), B^{\pm}(k))$, the energy eigenfunctions (31) are rewritten as follows:

$$\psi_m^{\pm}(x) = N_m^{\pm} \left[A^{\pm}(k_m^{\pm}) \, \mathrm{e}^{\mathrm{i}k_m^{\pm}x} + B^{\pm}(k_m^{\pm}) \, \mathrm{e}^{\mathrm{i}k_m^{\pm}(L-x)} \right], \tag{51}$$

where the normalization factors $N_m^{\pm}$ are given by

$$\left| N_m^{\pm} \right|^2 = \left(L + \left\{ A^{\pm}\left(k_m^{\pm}\right)\left[B^{\pm}\left(k_m^{\pm}\right)\right]^* + B^{\pm}\left(k_m^{\pm}\right)\left[A^{\pm}\left(k_m^{\pm}\right)\right]^* \right\} \frac{\mathrm{e}^{\mathrm{i}k_m^{\pm}L} - \mathrm{e}^{-\mathrm{i}k_m^{\pm}L}}{2\mathrm{i}k_m^{\pm}} \right)^{-1}. \tag{52}$$

With the help of the identities $A^{\pm}\left(k_m^{\pm}\right)\left[B^{\pm}\left(k_m^{\pm}\right)\right]^* + B^{\pm}\left(k_m^{\pm}\right)\left[A^{\pm}\left(k_m^{\pm}\right)\right]^* = \mathrm{tr}\left[\mathcal{P}_{\pm}\left(k_m^{\pm}\right)\sigma_1\right] = \pm\varepsilon_x\left(k_m^{\pm}\right), \mathrm{e}^{\mathrm{i}k_m^{\pm}L} - \mathrm{e}^{-\mathrm{i}k_m^{\pm}L} = -2\mathrm{i}\,\mathrm{Im}\left[s_{\pm}\left(k_m^{\pm}\right)\right]$ and (18), it is not difficult to show that the normalization factors can be written as

$$\left| N_m^{\pm} \right|^2 = \frac{1}{f_{\pm}'\left(k_m^{\pm}\right)}. \tag{53}$$

5. Feynman kernel

In this section, we prove our main goal of formula (3). To this end, let us first discuss the case of $0 < \alpha_{\pm} < 2\pi$. In the operator formalism, the Feynman kernel is then given by

$$K(x, T; x_0, 0) = \sum_{\xi=\pm} \sum_{m=1}^{\infty} \mathrm{e}^{-\mathrm{i}(k_m^{\xi})^2 T} \psi_m^{\xi}(x)\left[\psi_m^{\xi}(x_0)\right]^* + \text{(bound state contribution)}. \tag{54}$$

We note that a fake $k_0^+ = 0$ mode is not included in the above summation, as it should be. Substituting (51) into (54) and using relations (41), (45), (49a), (49b) and (53) for $0 < \alpha_{\pm} < 2\pi$, we can rewrite (54) as

$$K(x, T; x_0, 0) = \sum_{\xi=\pm} \sum_{k_m^{\xi} \in \sigma_{\xi}} \mathrm{e}^{-\mathrm{i}(k_m^{\xi})^2 T} \frac{1}{f_{\xi}'\left(k_m^{\xi}\right)}$$
$$\times \left\{ \left| A^{\xi}\left(k_m^{\xi}\right) \right|^2 \mathrm{e}^{\mathrm{i}k_m^{\xi}(x-x_0)} + \left[A^{\xi}\left(k_m^{\xi}\right)\right]^* B^{\xi}\left(k_m^{\xi}\right)\mathrm{e}^{\mathrm{i}k_m^{\xi}(L-x-x_0)} \right\}$$
$$+ \text{(bound state contribution)}. \tag{55}$$

We should note that the summations over $k_m^{\pm}$ can be enlarged to $\sigma_{\pm}$ and a fake $k_0^+ = 0$ mode is added in (55) with the relation $A^+\left(k_0^+\right) + B^+\left(k_0^+\right) = 0$. Now we can use the trace formula (46):

$$K(x, T; x_0, 0) = \sum_{\xi=\pm} \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} [s_{\xi}(p)]^n$$
$$\times \left\{ |A^{\xi}(p)|^2 \mathrm{e}^{\mathrm{i}T\left[p\left(\frac{nL+x-x_0}{T}\right)-p^2\right]} + [A^{\xi}(p)]^* B^{\xi}(p)\mathrm{e}^{\mathrm{i}T\left[p\left(\frac{(n+1)L-x-x_0}{T}\right)-p^2\right]} \right\}$$
$$+ \text{(bound state contribution)}, \tag{56}$$

where definitions (38) have been used. By use of relations (26), (48a), (49a), (49b), (50a)–(50d), we finally arrive at conclusion (3).

It is interesting to point out that the final expression (3) holds for other values of $\alpha_{\pm}$, even though relations (41), (45), (49a) and (49b) and the existence/nonexistence of a fake zero mode, as well as physical zero energy states, depend on $\alpha_{\pm}$. Furthermore, we emphasize that the knowledge of the energy eigenvalues and eigenstates is required in the expression of the Feynman kernel (54), while only the one-particle scattering matrix is sufficient to represent the Feynman kernel in our formulation. This suggests that expression (3) is more fundamental than the original one (54).

Before closing this section, we would like to briefly summarize the previous results. For the smooth subfamily of point interactions [6, 7], the boundary conditions are given by $\psi(L) = \mathrm{e}^{\mathrm{i}\theta}\psi(0)$ and $\psi'(L) = \mathrm{e}^{\mathrm{i}\theta}\psi'(0)$ $(0 \leqslant \theta < 2\pi)$, which corresponds to $e_z = 0$ and $(\alpha_+, \alpha_-) = (0, \pi)$ or $(\pi, 0)$, and the form of the Feynman kernel is well known [8, 9].

In the separated subfamily of point interactions [6, 7], the Neumann–Neumann, Dirichlet–Dirichlet, Neumann–Dirichlet, Dirichlet–Neumann boundary conditions at $x = 0$ and L, which correspond to $(\alpha_+, \alpha_-) = (0, 0), (\pi, \pi), (0, \pi), (\pi, 0)$ with $e_x = e_y = 0$, are vastly studied in the literature. For the scale-independent subfamily of point interactions [6, 7], the boundary conditions are given by $\psi(L) = e^{\rho+i\theta}\psi(0)$ and $\psi'(L) = e^{-\rho+i\theta}\psi'(0)$, which correspond to $(\alpha_+, \alpha_-) = (0, \pi)$ or $(\pi, 0)$ and arbitrary $\vec{e}$, and the kernel was obtained in [6, 7]. It is not difficult to verify that all of the results mentioned above coincide with our formula (3), although the exact forms of the kernels have been found so far only for a very limited class in the whole $U(2)$ parameter family. In fact, all the systems mentioned above have the common properties that make the analysis much easier: the infinite sequence of the momentum eigenvalues $k_m^\pm$ is equidistant and the normalization factors $N_m^\pm$ of the wavefunctions in (51) are independent of $k_m^\pm$, which implies that $\mathrm{d}f_\pm(k_m^\pm)/\mathrm{d}k$ are constant (see equation (53)) and hence formula (46) reduces to the Poisson summation formula. Our formula (3) holds true even when $k_m^\pm$ are given by transcendental roots and hence have no analytic expressions in the whole $U(2)$ parameter family. Thus, our results turn out to include the non-trivial extension of the previous works.

6. Conclusions

In this paper, we studied the particle propagation on a circle in the presence of a single point interaction compatible with the conservation of the probability current or the self-adjoint extension of the Laplace operator $-\mathrm{d}^2/\mathrm{d}x^2$ on $S^1\backslash\{0\}$. We uncovered the classical trajectories for a quantum particle on $S^1\backslash\{0\}$, which consist of 2^n distinct paths for a particle scattered n-times from the point interaction (point scatterer). We also illuminated a deep connection between the scattering theory on $\mathbb{R}\backslash\{0\}$ and the spectral property of $S^1\backslash\{0\}$, which, in roughly speaking, is summarized as the following correspondences:

eigenvalues of the S-matrix on $\mathbb{R}\backslash\{0\}$ $\Leftrightarrow$ energy spectrum of $S^1\backslash\{0\}$,

eigenvectors of the S-matrix on $\mathbb{R}\backslash\{0\}$ $\Leftrightarrow$ energy eigenfunctions on $S^1\backslash\{0\}$.

These correspondences may be regarded as some kind of an inverse scattering method [13] for singular zero-range potential in one spatial dimension. Indeed, once given an S-matrix on $\mathbb{R}\backslash\{0\}$ ('scattering data' on $\mathbb{R}$), we can construct the (positive) energy eigenfunctions as well as the (positive) energy spectrum on $S^1\backslash\{0\}$ via these correspondences.

We note that the eigenvalues of the S-matrix depend only on the three parameters α_+, α_- and e_x, whereas the eigenvectors depend on the full parameters of $U(2)$. The reason can be explained from the above correspondence because the energy spectrum of $S^1\backslash\{0\}$ depends only on the three parameters due to a $U(1)$ symmetry with respect to the parity in the spectral space [7].

The main success of this work is the systematic description of a one-particle Feynman kernel on a circle with a point interaction. The point is that we do not need any knowledge of the spectrum nor of the complete set of energy eigenfunctions of the system (except for the bound states). What we have to know is the classical trajectories of a particle and the one-particle scattering matrix.

Acknowledgments

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